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J. Phys. A: Math. Gen. **33** (2000) 5867–5878. Printed in the UK PII: S0305-4470(00)05068-X

Quasicrystals and Denjoy homeomorphisms

D J L Herrmann

Theoretische Fysica, Katholieke Universiteit Nijmegen, Postbus 9010, 6500 GL Nijmegen, The Netherlands

E-mail: `danielh@sci.kun.nl`

Received 9 June 1999, in final form 4 February 2000

Abstract. We assume that the atomic positions in a quasicrystal form a repetitive Delone set with a finite Bravais module. Therefore we investigate the dynamical system $(\Omega, \tau, \mathbb{R}^d)$ arising from the orbit closure of such a set. Using the cut-and-project method we construct a Poincaré section for the dynamical system $(\Omega, \tau, \mathbb{R}^d)$ such that the action of $\mathbb{R}^d$ reduces to an action of $\mathbb{Z}^d$. We obtain d commuting homeomorphisms $\varphi_1, \ldots, \varphi_d$ on a Cantor set X. In one dimension we relate (X, φ) to the support of the invariant measure of a homeomorphism on the circle (Denjoy homeomorphism). In this way we see that the K-groups with additional structure of the corresponding C^*-algebra classify these point sets and equivalences between different repetitive Delone sets are established. The discussion includes point sets with an acceptance domain given by a countable union of intervals or with a fractal atomic surface.

1. Introduction

Many successful concepts to describe transport properties of condensed matter systems are based on the fact that the underlying physical system can be seen as perfectly periodic. Although physicists have treated deviations as perturbations in many physical situations, there are systems like quasicrystals or amorphous systems which are intrinsically non-periodic. Furthermore, fundamental effects like the quantized Hall effect cannot be explained without the assumption of localized states, i.e., aperiodicity is no longer a perturbation of a perfect system. In [4, 5] a theory is developed extending concepts, like the Brillouin zone, for homogeneous but not necessarily periodic systems. For a crystal the Brillouin zone is topologically a torus and the physical observables belong to the algebra of continuous functions over the Brillouin zone tensored by the compact operators. Obviously, the topological nature of the Brillioun zone is induced by the atomic positions. In the general case the atomic positions give rise to a non-commutative C^*-algebra $\mathcal{A}$ [7] which is the generalization of the algebra of continuous functions over the Brillouin zone. However, there is no underlying topological space. One result of this theory is given by the gap labelling theorem. It identifies the possible values of the integrated density of states on a gap with the trace of the corresponding projection in $\mathcal{A}$. With the projections in $\mathcal{A}$ one constructs the K_0-group of $\mathcal{A}$. We are interested in a better understanding of its role for quasicrystals.

In section 2 we discuss the properties of a point set describing a quasicrystal and define strongly repetitive Delone sets. We use a topology on the set of point sets induced by a metric measuring the agreement of the point sets at the origin. Then the hull of a point set is defined as the dynamical system given by the orbit closure of the point set under the action of translations.

0305-4470/00/335867+12$30.00 © 2000 IOP Publishing Ltd

For the relation between the hull of a point set and the hull of a homogeneous system we refer to [7].

Due to the Poincaré construction, we have in one dimension a canonical way to pass from a continuous dynamical system to a discrete one, and vice versa. For a dynamical system $(\Omega, \tau, \mathbb{R}^d)$ we can also construct a Poincaré section, but the return points are in general no longer induced by a group action. For hulls of quasicrystals we construct in section 3 a Poincaré section such that the return points are induced by a $\mathbb{Z}^d$-action. Further we give a topological characterization of such hulls.

In section 4 we investigate strongly repetitive Delone sets in one dimension and characterize them by the K-groups (with additional structure) of their hulls. This result relies on the recent work by Giordano *et al* [15].

2. Quasicrystals and Delone sets

The geometric properties of a condensed matter system are given by the atomic positions. In the infinite volume limit we assume that the set of atomic positions $\mathcal{T} \subset \mathbb{R}^d$ is a Delone set, i.e. the set $\mathcal{T}$ fulfils

(1) $r_1 = \inf\{|x - y|, x, y \in \mathcal{T}\} > 0$,
(2) $r_2 = \sup\{s \in \mathbb{R}_+, \mathcal{T} \cap B_s(x) = \emptyset \text{ for } x \in \mathbb{R}^d\} < \infty$,

where $B_s(x) = \{y \in \mathbb{R}^d |, \|x - y\| < s\}$. We write $\mathcal{T} + x$ instead of $\tau_x(\mathcal{T})$, where τ is the canonical action of $\mathbb{R}^d$ induced by the translations.

We call $\mathcal{T}$ repetitive iff for every $s > 0$ there is an $t > 0$ such that every pattern of $\mathcal{T}$ contained in a ball of radius s appears in every pattern of $\mathcal{T}$ covering a ball of radius t. In other words every bounded pattern in $\mathcal{T}$ reoccurs in a relative dense way.

This concept of aperiodic order has been introduced in the context of quasicrystals by Danzer [10] and in the context of incommensurate structures by Aubry [2]. A periodic point set is uniquely determined, up to translation, by the occurring bounded patterns. This is in general not true for a repetitive Delone set (r-Delone set), i.e., there exist r-Delone sets consisting of the same bounded patterns which are not the same up to some translation. Actually, we will need a stronger condition on the distribution of bounded patterns in a r-Delone set.

Let $\mathcal{T}$ be an r-Delone set in $\mathbb{R}^d$. We call $\mathcal{T}$ strongly repetitive (or a strong r-Delone set) iff for every $s > 0$ there exists a monotonically decreasing function $d_s \in C_0(\mathbb{R}_+)$ such that for every pattern M satisfying $\emptyset \neq M \subset \mathcal{T} \cap B_s(x)$ for some $x \in \mathbb{R}^d$ there exists a $c_M > 0$ with

$$\left| c_M - \frac{\#\{y \in B_l(0) | M + y \subset \mathcal{T}\}}{\mathrm{vol} B_l(0)} \right| \leqslant d_s(l) \tag{1}$$

for all $l > s$. Then c_M is the frequency of the pattern M in $\mathcal{T}$. The uniform existence of all c_Ms is equivalent to the unique ergodicity of the hull defined below [31].

We define a metric on the space of all uniformly discrete point sets, in which two point patterns are close if they agree on a large ball about the origin. Here we follow the definition in [1]. Other equivalent metrics and topologies arise from other notions of distance [27–29,31]. In [7] a weaker topology is introduced, which is more appropriate from the physical point of view.

Let $\mathcal{T}$ and $\mathcal{S}$ be two uniformly discrete point sets in $\mathbb{R}^d$. Then we define a metric by $\boldsymbol{d}(\mathcal{T}, \mathcal{S}) = \inf(\{\varepsilon > 0 | \mathcal{T} + u \text{ and } \mathcal{S} + v \text{ agree on } B_{1/\varepsilon}(0) \text{ for some } \|u\|, \|v\| < \varepsilon\} \cup \{1/\sqrt{2}\})$. Here $\|\cdot\|$ is the usual norm on $\mathbb{R}^d$. Then we define $\Omega(\mathcal{T}) = \overline{\{\mathcal{T} + x | x \in \mathbb{R}^d\}}$ as the hull of $\mathcal{T}$ and consider in the following the dynamical system $(\Omega(\mathcal{T}), \tau, \mathbb{R}^d)$. One checks that τ is continuous.

Lemma 2.1. *Let $\mathcal{T}$ be an r-Delone set. Then every $\mathcal{S} \in \Omega(\mathcal{T})$ is also an r-Delone set. If $\mathcal{T}$ is strongly repetitive then so is $\mathcal{S} \in \Omega(\mathcal{T})$—with the same $d_s \in C_0(\mathbb{R}_+)$.*

Proof. Let $\mathcal{S} = \lim_{n\to\infty} \mathcal{T} + x_n$. Then the lemma is obtained by direct application of the definition of the metric and of (strong) repetition. □

A short calculation shows that the pattern frequencies c_M are actually constant over the hull. The following theorem is basically Gottschalk's theorem.

Theorem 2.2. *Let $\mathcal{T}$ be a Delone set. Then $\mathcal{T}$ is repetitive iff $\Omega(\mathcal{T})$ is a compact space and the hull of $\mathcal{T}$ is minimal.*

Proof. By the definition of the metric above, Ω is complete. Since $\mathcal{T}$ is repetitive, we have that Ω is precompact and therefore compact. Further, every $\mathcal{S} \in \Omega$ is also repetitive containing the same 'bounded point pattern' up to translation and therefore the orbit of $\mathcal{S}$ is dense. If Ω is compact then $\mathcal{S} \in \Omega$ has only a finite number of different patterns for each finite size. Since $(\Omega, \tau, \mathbb{R}^d)$ is minimal, every $\mathcal{S} \in \Omega$ contain 'all bounded patterns' and they are relatively dense. □

Actually, every Delone set has a compact hull if we use a weaker topology [7]. Let us remark that the answer to the question whether there exists a local rule which transforms one quasi-periodic tiling to another quasi-periodic tiling (like between the kits and darts tiling of Penrose and the Robinson–Penrose tiling) is related to the property of whether the corresponding dynamical systems are conjugated or not. Namely, the first one implies the second one and we believe also vice versa (maybe one needs to exclude some pathological cases).

3. The hull of an r-Delone set

Let $\mathcal{T}$ be an r-Delone set and let $\mathcal{B}$ be the Bravais module of $\mathcal{T}$ (the $\mathbb{Z}$-module generated by $\{x - y | x, y \in \mathcal{T}\}$). In the following we assume that $\mathcal{B}$ has finite rank. This is a reasonable assumption for quasicrystals from a physical point of view, since $\mathcal{T}$ is an idealization of a finite set of the atomic positions of a 'quasicrystal'. However, this is not necessarily true for incommensurate structures.

Under this assumption $\mathcal{T}$ can be written as the image under a projection of a subset in a higher-dimensional lattice. In this way the cut-and-project method comes into play. Its idea is to construct a specific point set as the image of a subset in a higher dimensional lattice [12,19,20] (more precisely, to construct in this way tilings of the space). Let us recall the fundamental ideas of this scheme [3,30]. It consists, by definition, of spaces and mappings:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{G} & \xrightarrow{\pi_2} & \mathbb{G} \\ & & \cup & & \\ & & L & & \end{array} \tag{2}$$

where $\mathbb{R}^d$ is a real Euclidean space and $\mathbb{G}$ some locally compact Abelian group, π_1 and π_2 are the projections onto them, and $L \subset \mathbb{R}^d \times \mathbb{G}$ is a lattice (L is a discrete subgroup such that $(\mathbb{R}^d \times \mathbb{G})/L$ is compact).

Further $\pi_1|_L$ is injective in $\mathbb{R}^d$ and $\pi_2(L)$ is dense in $\mathbb{G}$. One calls $\mathbb{R}^d$ (respectively $\mathbb{G}$) the physical (respectively internal) space. We assume that π_1 (respectively π_2) is the projection map on the first (respectively second) coordinate of $\mathbb{R}^d \times \mathbb{G}$. Therefore the setting of a cut-and-projection scheme is given by the triple $(\mathbb{R}^d, \mathbb{G}, L)$.

Remark. Cases where $\mathcal{B}$ has infinite rank can be treated by choosing a suitable locally compact group $\mathbb{G}$ [3]. For example, the vertices of the chair tiling [16] form an r-Delone set where $\mathcal{B}$ has infinite rank and can be described in this manner [3].

For a subset $A \subset \mathbb{G}$ we define a point set $\Lambda(A)$ in $\mathbb{R}^d$ by

$$\Lambda(A) = \{\pi_1(x) | x \in L, \pi_2(x) \in A\}. \tag{3}$$

A is called the acceptance domain or window of $\Lambda(A)$. In the literature [3, 23, 30] $\Lambda(A)$ is called a model set, iff $A = \overline{\text{int}(A)} \neq \emptyset$. A model set has many useful properties, especially the fact that it is strong repetitive if $\partial A \cap \pi_2(L) = \emptyset$.

We prefer not to impose such a condition directly on the acceptance domain. Therefore, we introduce another set in the internal space $\mathbb{G}$. Let $(\mathbb{R}^d, \mathbb{R}^n, L)$ be a cut-and-project scheme and $\mathcal{T}$ an r-Delone set with $\mathcal{T} \subset \pi_1(L)$. Then we call $\boldsymbol{PR}(\mathcal{T}) = \overline{\{\pi_2(x); x \in L \text{ with } \pi_1(x) \in \mathcal{T}\}}$ the projection range of $\mathcal{T}$. We say $\mathcal{T}$ is generated by $(\mathbb{R}^d, \mathbb{R}^n, L)$ if $\boldsymbol{PR}(\mathcal{T})$ is compact. For model sets the projection range agrees with the acceptance domain. However, we remark that even in the hull of a model set we find r-Delone sets which are not model sets, namely the *singular* one. (The projection range is the closure of the atomic surface [21] translated to the internal space.)

Lemma 3.1. *Let $\mathcal{T}$ be an r-Delone set generated by a cut-and-project scheme $(\mathbb{R}^d, \mathbb{R}^n, L)$ and Ω its hull. Then all $\mathcal{S} \in \Omega$ with $\mathcal{S} \subset \pi_1(L)$ have the same projection range up to translation.*

In the following we construct a covering space for the hull of an r-Delone set and characterize in this way its topological structure. Let $\mathcal{T}$ be an r-Delone set generated by $(\mathbb{R}^d, \mathbb{R}^n, L)$. Let P_L be the set of non-empty subsets of L with the topology induced by the product topology of $\{0, 1\}^L$ when identified in the canonical way. Then the topology of P_L is given by the basis of open sets $(A, B) = \{N \in P_L | A \subset N \text{ and } B \cap N = \emptyset\}$ with $A \neq \emptyset$ and B are finite subsets of L. Therefore P_L is a locally compact and totally disconnected space and in itself dense.

Lemma 3.2. *Let $\mathcal{T}$ be an r-Delone set generated by a cut-and-project scheme $(\mathbb{R}^d, \mathbb{R}^n, L)$. Then $P = \{N \in P_L, \pi_1(N) \in \Omega(\mathcal{T})\}$ is a locally compact and totally disconnected space and in itself dense.*

Proof. π_1 is a continuous map and $\Omega(\mathcal{T})$ compact so that P is closed and therefore locally compact and totally disconnected. Suppose $N \in P$ is an isolated point, then there exists an open set $(A, B) \subset P_L$ with $(A, B) \cap P = \{N\}$. Since $\pi_1(N)$ is repetitive we find $g \in L$ such that $N + g \in (A, B) \cap P$ in contradiction to $(A, B) \cap P = \{N\}$. □

Next, we describe Ω as a twisted product space of a Cantor set and a $\mathbb{Z}^d$-action. The lattice L induces a canonical action on $P \times \mathbb{R}^d$ via $S_g(N, x) = (N - g, x - \pi_1(g))$.

Proposition 3.3. *We have the following commuting diagram:*

$$\begin{array}{ccc} P \times \mathbb{R}^d & \xrightarrow{h} & \mathbb{R}^n \times \mathbb{R}^d \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle q} \\ \Omega & \xrightarrow{h_*} & \mathbb{T}^n \times \mathbb{T}^d \end{array} \tag{4}$$

with continuous maps defined below.

Proof. Let $\phi(N, x) = \pi_1(N) - x$. Then ϕ is continuous and surjective with $\phi(N, x) = \phi(N', x')$ if and only if there is an element $g \in L$ with $S_g(N, x) = (N', x')$. Therefore we can identify Ω with $P \times \mathbb{R}^d/_L$. We fix $M \in P$. Due to lemma 3.1 there exists a unique translation such that $\overline{\pi_2(N)} + y = \overline{\pi_2(M)}$. We define the continuous map h by $h(N, x) = (y, x)$. Next we define q as the quotient map induced by taking the quotient by $L \subset \mathbb{R}^n \times \mathbb{R}^d$. For $\mathcal{S} \in \Omega$ if we choose $(N, y) \in P \times \mathbb{R}^d$ with $\phi(N, y) = \mathcal{S}$ and define $h_*(\mathcal{S}) = q \circ h(N, y)$, one easily verifies that $h_*(\mathcal{S})$ is independent of the choice of (N, y) and h_* is continuous. □

The map h identifies elements with the same projection range and there exist at least two different elements with the same projection range if the boundary of their projection ranges have non-empty intersection with $\pi_2(L)$. For model sets these elements are called *singular*.

Lemma 3.2 shows that P as a space is topologically the same for all r-Delone sets and therefore the specific properties of Ω are determined by the action S_g, $g \in L$. We use this fact to reduce the continuous dynamical system $(\Omega, \mathbb{R}^d)$ to a discrete one.

Theorem 3.4. *Let $V, W \subset \mathbb{R}^n \times \mathbb{R}^d$ be such that $V \cong \pi_2(V) \cong \mathbb{R}^n$, $V \oplus W = \mathbb{R}^n \times \mathbb{R}^d$ and $(V \cap L) \oplus (W \cap L) = L_{\mathrm{v}} \oplus L_{\mathrm{w}} = L$. Then we have a commuting diagram*

$$\begin{array}{ccc} P \times \mathbb{R}^d & \xrightarrow{q_{\mathrm{v}}} & X \times \mathbb{R}^d \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle q_{\mathrm{w}}} \\ \Omega & \cong & X \times \mathbb{R}^d/_\sim \end{array} \tag{5}$$

with spaces and continuous maps as defined below.

Proof. Clearly, we have a group action S_g of L on $P \times \mathbb{R}^d$. We consider $X \times \mathbb{R}^d = h^{-1}(V)/_{L_{\mathrm{v}}} \times \mathbb{R}^d \cong P \times \mathbb{R}^d/_{L_{\mathrm{v}}}$ with h defined in the commuting diagram (4). For every $N \in P$ there exists a unique $z \in \mathbb{R}^d$ such that $h(N, z) \in V$. We define the map $q_{\mathrm{v}} : (M, x) \to ([(M, z)]_{L_{\mathrm{v}}}, x - z)$, which is continuous. Let $\sim$ be the equivalence relation induced by $P \times \mathbb{R}^d/_L \cong X \times \mathbb{R}^d/_\sim$ and let q_{w} be the quotient map. Then q_{w} is obviously continuous. □

Corollary 3.5. *X is a Cantor set and $\sim$ induces a $\mathbb{Z}^d$ action φ on X, i.e. Ω is a twisted product space of a Cantor set with a $\mathbb{Z}^d$ action. Further the complex K-group $K^n(\Omega)$ is naturally isomorphic to the K-group $K_{n+d}(\mathcal{A})$ of C^*-algebras, where $\mathcal{A} = C(X) \times_{\varphi} \mathbb{Z}^d$ or $\mathcal{A} = C(\Omega) \times \mathbb{R}^d$.*

Proof. Since L_v is a lattice in V and $P \cong h^{-1}(V) = \bigcup_{g \in L_v} S_g(A)$ for some compact set $A \subset P$, we have that X is a compact, totally disconnected set with no isolated point due to lemma 3.2. We show for fixed (N, z) with $h(N, z) = (y, z) \in V$ that $K = \{x \in \mathbb{R}^d; \exists g \in L : h \circ S_g(N, z + x) \in V\}$ is a lattice. Let $x_1, x_2 \in K$. Then there exist $g_1, g_2 \in L$ with $h \circ S_{g_i}(N, z + x_i) = h(N - g_i, z + x_i - \pi_1(g_i)) = (y - \pi_2(g_i), z + x_i - \pi_1(g_i)) \in V$ $(i = 1, 2)$. Since $h \circ S_{g_1+g_2}(N, z+x_1+x_2) = h(N - g_1 - g_2, z+x_1+x_2 - \pi_1(g_1+g_2))(y - \pi_2(g_1+g_2), z+x_1+x_2 - \pi_1(g_1+g_2)) = (y - \pi_2(g_1), z+x_1 - \pi_1(g_1)) + (y - \pi_2(g_2), z+x_2 - \pi_1(g_2)) - (y, z) \in V$, we have $x_1 + x_2 \in K$. Since K is relative dense and $K_g = \{x \in \mathbb{R}^d; h \circ S_g(N, z + x) \in V\} = \{0\}$ for $g \in L_v$, we have that $\sim$ induces a $\mathbb{Z}^d$ action.

For the theory and notation of K-groups we refer the reader to [8]. By Connes' Thom isomorphism we have $K^n(\Omega) \cong K_{n+d}(C(\Omega) \times \mathbb{R}^d)$. $C(X) \times_{\varphi} \mathbb{Z}^d$ is the groupoid C^*-algebra corresponding to the smooth transversal X for $(\Omega, \mathbb{R}^d)$ and therefore $C(X) \times_{\varphi} \mathbb{Z}^d$ is strongly Morita equivalent to $C(\Omega) \times \mathbb{R}^d$ implying the same K-groups. □

We call $(X, \varphi, \mathbb{Z}^d)$ the Cantor system of the hull $(\Omega, \tau, \mathbb{R}^d)$ (depending on V and W). h_* of theorem 3.4 induces a continuous surjective map $\hat{h} : X \to \mathbb{T}^n$ and one easily sees that $\hat{h}$ is a semi-conjugation to $(\mathbb{T}^n, R_1, \ldots, R_d)$, where $R_1, \ldots, R_d$ is a set of rigid rotation on $\mathbb{T}^n$.

Remark. Let us consider a Schrödinger operator H with a potential constructed by atomic potentials located on an r-Delone set $\mathcal{T}$. Then a smooth transversal in $\Omega(\mathcal{T})$ corresponds to a specific construction of a tight-binding Hamiltonian for H [7]. The above construction of the smooth transversal corresponds to an effective Hamiltonian on a subspace of $L^2(\mathbb{R}^d)$ which is invariant under a subgroup of translations (a lattice).

4. Denjoy homeomorphisms and one-dimensional model sets

In the context of the gap labelling theorems for one-dimensional discrete Schrödinger operators with various types of potentials (potentials taking finitely many values, limit periodic potentials coming from automatic sequences, Kohmoto model and B-S model) the image of the corresponding K-groups under the trace have been calculated [5, 6]. All these potentials correspond to specific types of Delone sets (not necessarily r-Delone sets). On the other hand, in a recent work [15] Cantor systems $(X, \varphi, \mathbb{Z})$, i.e. X is a Cantor set and $\mathbb{Z}$ acts minimally on X, are classified by their K-groups (with additional structure). Our analysis is based on this work. It turns out that the hulls of strong r-Delone sets are characterized by Denjoy systems and odometer systems. Denjoy systems correspond to cut-and-project schemes with internal space $\mathbb{R}$ and odometer systems correspond to cut-and-project schemes with internal space a locally compact group $\mathbb{G} \ncong \mathbb{R}$ which will be presented elsewhere [18].

Denjoy system. The homeomorphisms of the circle with no periodic orbits are classified by a set of invariants [11, 17, 22, 24]. We review briefly the setting to fix notation; see also [25], sections 1–3.

Rotation number. We write the unit circle as $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1]/_{0\sim 1}$ with orientation induced by $\mathbb{R}$ and φ an orientation-preserving homeomorphism of $\mathbb{T}$. Then φ can be 'lifted' to a strictly increasing continuous function $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ which satisfies $\tilde{\varphi}(x + 1) = \tilde{\varphi}(x) + 1$. The lifting is unique if we impose $0 \leqslant \tilde{\varphi}(0) < 1$. The limit

$$\lim_{n\to\infty} \frac{\tilde{\varphi}^n(x)}{n}$$

exists, is independent of $x \in \mathbb{R}$ and lies in the interval [0, 1]. The rotation number $\rho(\varphi)$ of φ is defined by this limit modulo 1. For example, if φ is the rigid rotation $R_\theta : t \to t + \theta (\text{mod } 1)$, then the rotation number is $\rho(R_\theta) = \theta$, i.e. the average rotation of a point. The rotation number $\rho(\varphi)$ is rational if and only if φ has a periodic orbit. In particular $\rho(\varphi) = 0$ if and only if φ has a fixed point. For further properties of the rotation number see [9, 17].

Theorem 4.1. [24, 25] *Let φ be a homeomorphism of the circle $\mathbb{T}$ with no periodic orbits, and let $\theta = \rho(\varphi)$ be the irrational rotation number of φ. Let x be any point of $\mathbb{T}$. Then the points $x_n = \varphi^n(x)$ are placed on $\mathbb{T}$ in the same order as the points $y_n = n\theta (\text{mod } 1)$, $n \in \mathbb{Z}$.*

Further there exists a continuous surjective map $h : \mathbb{T} \to \mathbb{T}$ so that

$$h \circ \varphi = R_\theta \circ h \tag{6}$$

where $R_\theta(t) = t + \theta (\text{mod } 1)$, i.e. $(\mathbb{T}, \varphi)$ is semi-conjugated to $(\mathbb{T}, R_\theta)$. Moreover h in (6) is unique up to a rotation. Also, φ is uniquely ergodic, i.e., there exists a unique φ invariant probability measure μ on $\mathbb{T}$. In fact $\mu = \mathrm{d}h$ and so, in particular, $\mu([a, b]) = h(b) - h(a)$, where $0 \leqslant a < b < 1$.

A Denjoy homeomorphism is a homeomorphism φ of $\mathbb{T}$ with no periodic orbits such that φ is not conjugated to a rigid rotation, i.e., $\rho(\varphi) = \theta$ is irrational and φ is not conjugated to R_θ. Let μ be the unique invariant measure $\mu = \mathrm{d}h$ of theorem 4.1 and $\Sigma = \text{support}(\mu)$. Then Σ is the only minimal invariant set of $(\mathbb{T}, \varphi)$ and Σ is a Cantor set (totally disconnected compact set with no isolated point). We can write Σ as $\Sigma = \mathbb{T} \setminus \bigcup_{n=1}^{\infty} I_n$, where $\bigcup_{n=1}^{\infty} I_n$ is a countable disjoint union of open intervals, the intervals $I_1, I_2, I_3, \ldots$ being the components of $\mathbb{T} \setminus \Sigma$. The map h in (6) collapses each interval $I_n = (a_n, b_n)$ into a single point. We call the countable set $\{a_n, b_n | n \in \mathbb{N}\}$ the *accessible* points of Σ. The accessible points pair naturally two-by-two by being end-points of disjoint components in Σ. Also h is one-to-one on the inaccessible points $\Sigma \setminus \{a_n, b_n | n = 1, 2, \ldots\}$. We set

$$Q(\varphi) = \{h(x) | x \text{ accessible point of } \Sigma\} = \{h(I_n) | n \in \mathbb{Z}\}. \tag{7}$$

$Q(\varphi)$ is uniquely determined by φ up to a rigid rotation. The set $Q(\varphi)$ is countable and invariant under R_θ. Therefore one can choose $x_0, x_1, x_2, \ldots \in Q(\varphi)$ with disjoint orbits $O_n = \{R_\theta^k(x_n) | k \in \mathbb{Z}\}$ such that

$$Q(\varphi) = \coprod_{n \in A} O_n \tag{8}$$

where $\coprod$ is the disjoint union and A at most countable. A Denjoy system (Σ, φ) is given by a Denjoy homeomorphism $\hat{\varphi}$ restricted to the support of its invariant measure.

Theorem 4.2. [22, 25] *Two Denjoy homeomorphisms φ_1 and φ_2 are conjugate via an orientation-preserving homeomorphism if and only if $\rho(\varphi_1) = \rho(\varphi_2)$ and $Q(\varphi_1) = R_\beta(Q(\varphi_2))$ for some $\beta \in [0, 1)$.*

Theorem 4.3. [25] *Let φ be a Denjoy homeomorphism with $\rho(\varphi) = \theta$ and let Σ be the unique minimal invariant Cantor set. Let D_φ be the simple C^*-algebra $C(\Sigma) \times_\varphi \mathbb{Z}$ with unique (faithful) normalized trace $\hat{\mathrm{tr}}$. Then $K_0(D_\varphi) = \bigoplus_1^{n(\varphi)} \mathbb{Z}$ and $K_1(D_\varphi) = \mathbb{Z}$. Moreover, the range of $\hat{\mathrm{tr}}$ on the projections in D_φ is $(\mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\gamma_2 + \cdots + \mathbb{Z}\gamma_{n(\varphi)}) \cap [0, 1]$. In particular, if $1, \theta, \gamma_2, \ldots, \gamma_{n(\varphi)}$ are linearly independent over the rational numbers, then*

$$\hat{\mathrm{tr}}_* : K_0(D_\varphi) \to \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\gamma_2 + \cdots + \mathbb{Z}\gamma_{n(\varphi)} \tag{9}$$

is an order isomorphism of ordered groups, where $\mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\gamma_2 + \cdots + \mathbb{Z}\gamma_{n(\varphi)}$ inherits the order of $\mathbb{R}$ and $\hat{\mathrm{tr}}_*$ is the induced homomorphism.

The proofs consist of solving Pimsner–Voiculesco six-terms exact sequences obtained from the short exact sequences; we refer to the recent work by Putnam *et al* [25] for details.

Let $\mathcal{T} = \Lambda(A)$ be generated by the cut-and-project scheme $(\mathbb{R}, \mathbb{R}, L)$ with an acceptance domain $A = \cup_{k=0}^{n} [a_k, b_k]$ and $\partial A \cap \pi_2(L) = \emptyset$. $\mathcal{T}$ is a generic model set and therefore a strong r-Delone set. We may assume that L is a square lattice, since $\pi_1(L)$ is only relevant for Ω.

Theorem 4.4. *Let (X, φ) be the Cantor system of $\Omega(\mathcal{T})$ in corollary 3.5 with $V = \mathbb{R}e_1$ and $W = \mathbb{R}e_2$ and e_1, e_2 a basis of L. Then $(X, \varphi, \mathbb{Z})$ is conjugated to a Denjoy system (Σ, ψ) with rotation number $\rho(\psi) = \pi_1(e_1)/\pi_1(e_2)$ and $Q(\psi)$ given below.*

Proof. Let $\hat{h} : X \to V \bmod \mathbb{Z}e_1 \cong \mathbb{T}$ be the continuous map induced by h of theorem 3.3. Then $\hat{h}$ is a semi-conjugation of (X, φ) with $(\mathbb{T}, R_\theta)$ with $\theta = \pi_1(e_1)/\pi_1(e_2)$ and is injective on $\mathbb{T} \setminus Q(\psi)$, with $Q(\psi) = \{h_*(\mathcal{S}) | \partial \boldsymbol{PR}(\mathcal{S}) \cap \pi_2(L) \neq \emptyset\}$ since h_* identifies exactly those elements which have the same projection range. More precisely h_* identifies two elements on $h_*^{-1}(Q(\psi))$ corresponding to the right and left limit. Hence $X \cong \mathbb{T} \setminus \bigcup_{k=1}^{s} \bigcup_{n=1}^{\infty} I_n^s$ and φ induces a conjugated Denjoy homeomorphism on $\mathbb{T}$ in the obvious way. □

Remark. Since h_* identifies precisely the singular model set pairs (the two model sets in such a pair differ only on a finite region, since the boundary of the acceptance domain is finite) in the transversal X, the dimension of the K_0-group is given by the number of singular r-Delone set pairs orbits and the order structure of the K_0-group is given by the distance in the internal space.

Now we consider the general situation for an r-Delone set in one dimension. For dynamical systems there exist, beside topological conjugacy, other types of equivalence relations.

Orbit equivalence. Two dynamical systems (X_1, φ_1) and (X_2, φ_2) are *(topologically) orbit equivalent* if there exists a homeomorphism $F : X_1 \to X_2$ such that $F(\mathrm{orbit}_{\varphi_1}(x)) = \mathrm{orbit}_{\varphi_2}(F(x))$ for all $x \in X_1$. We call F an *orbit map.* For every point $x \in X_1$ there exists an integer $n(x)$ such that $F \circ \varphi_1(x) = \varphi_2^{n(x)} \circ F(x)$. Likewise, there exists an integer $m(x)$ such that $F \circ \varphi_1^{m(x)}(x) = \varphi_2 \circ F(x)$. If (X_1, φ_1) (and hence (X_2, φ_2)) is minimal it is easily seen that m and n are uniquely defined integer-valued functions on X_1. We call m and n the *orbit co-cycles* associated to the orbit map F. One easily verifies that orbit equivalence is an equivalence relation.

Strong orbit equivalence. Two dynamical systems (X_1, φ_1) and (X_2, φ_2) are *strongly orbit equivalent* if there exists an orbit map $F : X_1 \to X_2$ such that the orbit co-cycles n and m associated to F have at most one point of discontinuity. Due to theorem 2.1 in [15] we know that strong orbit equivalence is an equivalence relation.

For a Cantor system (X, φ), i.e. X is a Cantor set and $\mathbb{Z}$ acts minimal on X, in [15] these equivalence relations are related to the ordered K-theory of $C^*(X, \varphi)$. We use these results to obtain a characterization of a strong r-Delone set.

Theorem 4.5. *Let $\mathcal{T}$ be a strong r-Delone set in $\mathbb{R}$. Then $Y = \{\mathcal{S} \in \Omega(\mathcal{T}), 0 \in \mathcal{S}\}$ is a smooth transversal of $(\Omega(\mathcal{T}), \tau, \mathbb{R})$. Let φ be the first return map for Y. Then (Y, φ) is orbit equivalent to a Denjoy system or an odometer system.*

Corollary 4.6. *Let $\mathcal{T}$ be a strong r-Delone set in $\mathbb{R}$ coming from a cut-and-project scheme $(\mathbb{R}, \mathbb{R}^n, L)$ and (X, φ) is the Cantor system of corollary 3.5. Then (X, φ) is orbit equivalent to a Denjoy system.*

Proof. (Theorem 4.5) Let $\mathcal{S} \in \Omega(\mathcal{T})$. Then $\inf\{d(\mathcal{S}, \mathcal{S}'), \mathcal{S}' \in Y\} > \inf\{||x||, x \in \mathcal{S}\}$ and therefore Y is closed and hence compact. Since $\mathcal{T}$ is repetitive we have that the topology of Y is generated by a set of open and closed bases (hence Y is totally disconnected) and $(\Omega, \tau, \mathbb{R})$ is minimal by theorem 2.2 and so (Y, ϕ) is also minimal. Therefore (Y, ϕ) is a Cantor system.

We show that for every $f \in C(Y)$ the sequence $1/N \sum_{k=0}^{N-1} f \circ \varphi$ converges uniformly on Y to a constant. Let $\varepsilon > 0$ and $a \in Y$. We define $\chi_a^\varepsilon(x) = 1$ if $d(x, a) < \varepsilon$ and $\chi_a^\varepsilon(x) = 0$ elsewhere. Then $\chi_a^\varepsilon(x) \in C(Y)$ and all such χ_a^ε form a dense subalgebra in $C(Y)$. By theorem 2.1 $x \in Y$ is strongly repetitive and therefore there exists a function $d_s \in C_0(\mathbb{R})$ and $c > 0$ such that $|1/N \sum_{k=0}^{N-1} \chi_a^\varepsilon(x) \circ \varphi(x) - c| \leqslant d_{1/\varepsilon}(Nr_1)$. Therefore $1/N \sum_{k=0}^{N-1} \chi_a^\varepsilon(x) \circ \varphi \to c$ uniformly in x. The functions χ_a^ε are dense in $C(Y)$ and therefore every $f \in C(Y)$ converges uniformly to a constant. This limit defines a linear functional L on $C(Y)$ and therefore a φ-invariant measure $\mu \in M_\varphi(Y)$. Since $\{x \in Y | \lim_{N\to\infty} 1/N \sum_{k=0}^{N-1} f(x) \circ \varphi = \int f \,\mathrm{d}\mu\} = Y$ there exists no other φ-invariant measure, i.e. (Y, φ) is uniquely ergodic.

By corollary 2 of [15] every uniquely ergodic Cantor system is either orbit equivalent to a Denjoy system or an odometer system. □

Proof. (Corollary 4.6) Analogously to theorem 4.5 we obtain that (X, φ) is either orbit equivalent to a Denjoy system or an odometer system. Since $C^*(X, \varphi)$ contains a subalgebra isomorphic to a rotation algebra A_α with irrational α, again by corollary 2 of [15] it cannot be an odometer system. □

Remark. We believe that theorem 4.5 does not depend on the particular form of the chosen smooth transversal. Consider, for example, the smooth transversals given by $Y_s = \{\mathcal{S} \in \Omega(\mathcal{T}), \mathcal{S} \cup B_s(0) = \mathcal{T} \cup B_s(0)\}$ for large enough $s > 0$. Then theorem 4.5 holds also.

5. Examples

The following examples show how theorem 4.5 can be used to establish (strong) orbit equivalence between hulls of different r-Delone sets.

Every Delone set in one dimension can be described by a (two- sided) infinite word with letters indicating the distance between two consecutive points (with possibly infinite number of different letters). On the other hand, a (two-sided) infinite word defines, up to translation, a Delone set if we assign to each letter a distance (if the alphabet is infinite the distances must have a strict positive upper and lower bound). Every primitive substitution rule on a finite alphabet defines a (two-sided) infinite word. The corresponding Delone set is strong repetitive and there is a natural choice for the assigned distance to each letter such that the substitution induces a self-similarity. For a more detailed description and discussion, refer to [6, 21, 26].

One of the most studied substitution sequences is the Fibonacci sequence. It is a substitution σ_F on two letters A and B with

$$\sigma_F(A) = AB \qquad \text{and} \qquad \sigma_F(B) = A. \tag{10}$$

The substitution matrix encodes the number of letters A and B in the words $\sigma_F(A)$ and $\sigma_F(B)$ respectively:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

However, the order of the letters in which they occur is not encoded. The sequence $A_n = \sigma_F^{2n}(A), \sigma_F^{2n}(A)$ converges towards an infinite sequence w, such that $\sigma_F(w) = w$, where ',' denotes the zero point (we take the square of σ_F, since otherwise we end up with an oscillation between two words). Now we assign to w an infinite sequence of points in a two-dimensional lattice L (for a substitution on n letters we would consider an n-dimensional lattice). Let $\{e_1, e_2\}$ be a basis of L. Then we define $x_0 = 0$ and the kth point x_k so that the vector difference

$$l_k = x_{k+1} - x_k$$

is chosen according to $l_k = e_1$ (respectively $l_k = e_2$) if the kth letter of w is A (respectively B). We have therefore

$$x_k = m^A(k)e_1 + m^B(k)e_2$$

where the integers $m^A(k)$ and $m^B(k)$ are, respectively, the numbers of letters A and B among the first k letters of the sequence w. We obtain an infinite staircase-shaped broken line if we connect x_k with x_{k+1}, drawn on the lattice, which escapes to infinity along the mean direction of the vector

$$v = \lim_{k\to\infty} \frac{x_k}{k} = \rho^A e_1 + \rho^B e_2 \tag{11}$$

where ρ^A (respectively ρ^B) is the frequency of A (respectively B) in w, which can be calculated from the substitution matrix M. In this way the cut-and-project formalism $(\mathbb{R}, \mathbb{R}, L)$ comes into play. The physical space is spanned by v in (11). Then the acceptance domain is given by $[\pi_2(e_1), \pi_2(e_2)[$ and the r-Delone set is given by $\Lambda([\pi_2(e_1), \pi_2(e_2)[) = \pi_1(\{x_k, k \in \mathbb{Z}\})$.

The first example considers a substitution sequence with the same substitution matrix as the square of the Fibonacci substitution, but the order of the letters are interchanged:

$$\sigma(A) = AAB \qquad \text{and} \qquad \sigma(B) = BA. \tag{12}$$

Again the sequence $A_n = \sigma^{2n}(A), \sigma^{2n}(A)$ converges towards an infinite sequence w, such that $\sigma_F(w) = w$. The word defines an r-Delone set $\mathcal{T}$ in the same way as above. Let us remark that $\mathcal{T}$ has a fractal atomic surface and the Fourier amplitudes have a different behaviour than in the Fibonacci case [21, figure 7].

Let us consider the Cantor system $(X_\sigma, \varphi_\sigma)$ corresponding to the hull of $\mathcal{T}$ (corollary 3.5). Following [6] we calculate $\hat{\mathrm{tr}}_* K_0(C^*(X_\sigma, \varphi_\sigma)) = \mathbb{Z} + \tau\mathbb{Z} + \tau/2\,\mathbb{Z}$, where τ is the golden mean. Therefore $(X_\sigma, \varphi_\sigma)$ is not orbit equivalent to the hull of the Fibonacci sequence. However, $(X_\sigma, \varphi_\sigma)$ is orbit equivalent to the Denjoy system (Σ, φ) with $Q(\varphi) = \{(\mathbb{Z} + \tau\mathbb{Z}) \bmod 1\} \cup \{\tau/2 + (\mathbb{Z} + \tau\mathbb{Z}) \bmod 1\}$. In addition, we obtain that the dimension of $K_0(C^*(X_\sigma, \varphi_\sigma))$ is three using the self-similarity induced by the substitution [1] and therefore $K_0(C^*(X_\sigma, \varphi_\sigma)$ has no infinitesimal element. So the two systems are strong orbit equivalent [15] which is equivalent to $C^*(X_\sigma, \varphi_\sigma) \stackrel{\sim}{=} C^*(\Sigma, \varphi)$. The Denjoy system (Σ, φ) can be interpreted as the Cantor system of two coupled Fibonacci sequences. Therefore (ordered) K-theory establishes relations between physically different systems and the question arises of which other physical properties agree, except for the possible values of the integrated densities of states on the gaps.

Now we compare strong orbit equivalent Denjoy systems. Let $0 < \gamma_0, \ldots, \gamma_n < 1$ be rationally independent, then $(G, G_+, 1)$ with $G = \mathbb{Z} + \mathbb{Z}\gamma_0 + \cdots + \mathbb{Z}\gamma_n \subset \mathbb{R}$ and $G_+ = G \cap \mathbb{R}_+$ is a simple dimension group. Let (Σ_k, φ_k) be the Denjoy system with rotation number $\rho(\varphi_k) = \gamma_k$ and invariant $Q(\varphi_k) = \bigcup_{l \neq k} \{\gamma_l + (\mathbb{Z} + \mathbb{Z}\gamma_k) \mod 1\}$. Due to theorem 4.3 the K_0-group of $C^*(\Sigma_k, \varphi_k)$ is $(G, G_+, 1)$. Furthermore, the Denjoy systems (Σ_k, φ_k) are pairwise non-conjugate. By theorem 2.2 in [15] they are all strong orbit equivalent and therefore all $C^*(\Sigma_k, \varphi_k)$ are isomorphic. According to theorem 4.4, we can interpret each Denjoy system as the Cantor system of a hull $(\Omega(\mathcal{T}), \tau, \mathbb{R})$ where $\mathcal{T}$ is an r-Delone set generated by a cut-and-project scheme with $\gamma_k = \pi_1(e_1)/\pi_1(e_2)$ and acceptance domain $A = \bigcup_{l=0}^{n} [\pi_2(e_2), \pi_2(e_1)[+x_l$ with $x_k = 0$ and $x_l = \gamma_l$ otherwise. Physically, we can interpret $\mathcal{T}$ as $n+1$-coupled r-Delone sets, each of them corresponding to one interval. The generators of the K_0-group have two different origins. One generator 'measures the average translation in internal space' and the other 'measures the distances' between orbits of different pairs of singular r-Delone sets.

Let $\mathcal{T} = \Lambda(A)$ be generated by the cut-and-project scheme $(\mathbb{R}, \mathbb{R}^n, L)$ with an acceptance domain $A = \overline{\mathrm{int} A}$ and $\partial A \cap \pi_2(L) = \emptyset$. Then $\mathcal{T}$ is a strong r-Delone set. Let μ be the unique ergodic measure on the corresponding Cantor system (X, φ). Then every Denjoy system (Σ, ψ) with $Q(\psi) = \{\mu(m) | m$ is clopen set in $X\}$ is orbit equivalent to (X, φ) (corollary 4.6).

6. Remarks and conclusion

For an r-Delone set generated by the cut-and-project method we obtained a topological description of the hull. It is given by a torus cut along the orbits of singular r-Delone sets if we are dealing with model sets with polygonal acceptance domains. We have constructed a smooth transversal X of the hull $(\Omega, \tau, \mathbb{R}^d)$ such that the $\mathbb{R}^d$-action reduces to a $\mathbb{Z}^d$-action. Since the corresponding K-groups agree, this is a starting point for the calculation of the K-groups for r-Delone sets in higher dimensions [13] (during the preparation of this article the work of [14] came to our attention).

In one dimension, r-Delone sets are characterized by their ordered K-groups with distinguished order element. For strong r-Delone sets we obtain new 'equivalences' between hulls of r-Delone sets and relate them to Denjoy systems. Here further work is required to determine the consequences. For the distinction of different Denjoy systems the order structure of the K_0-group is crucial, therefore we expect that, in general, the additional structure of the K-groups will be relevant. The K_0-groups of systems which are orbit equivalent, but not strongly orbit equivalent, differ in the infinitesimal subgroup. It would be interesting to find such a situation and to understand the role of the infinitesimal elements.

Acknowledgments

The author is grateful to T Janssen, J Bellissard and M Baake for interesting discussions. The *Studienstiftung des deutschen Volkes* is acknowledged for partial financial support.

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